GCSE Maths – Number

Powers, Roots and Fractional Indices

Notes

WORKSHEET

Powers and Roots

A power is a number that tells another number and (called the **base**) how many times it needs to **multiply by itself**. The power is written as a small floating number to the top right of the base.

Examples: Writing a power

$$10^2 = 10 \times 10 = 100$$

$$5^3 = 5 \times 5 \times 5 = 125$$

A power is also called an **index**, where the plural of index is **indices**. These can be whole numbers (**integers**), **fractions**, **negative numbers** or a **combination** of these.

The opposite of a power is called a root. Just like with powers, we can have different types of roots such as square roots and cube roots. These are written with a **root symbol**: $\sqrt{\ }$
Depending on which root is being calculated, a number is written on the start of the root:

Square root: $\sqrt{\ }$
Cube root: $\sqrt[3]{\ }$
Fourth-root: $\sqrt[4]{\ }$

Examples: Writing a root

$$\sqrt{81} = \pm 9$$

Both +9 and - 9 are valid square roots because $(+9)^2 = (-9)^2 = 81$.

$$\sqrt[3]{27} = 3$$

3 is the only valid cube root here.

$$(\sqrt{36})^2 = (6)^2 = 36$$

Squaring a square root leads to a cancellation.

Sometimes we cannot simplify a root, so we can leave it as it is. This form is called **surd form**. There are also cases where we can partly **simplify** a root - sometimes we obtain an integer but other times the simplified answer is also left in **surd form**.

Examples: Surd form and partly simplified roots

$$\sqrt{49} = \sqrt{(7^2)} = 7$$

$\sqrt{17}$ Cannot be simplified so we leave it as it is

$$\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3 \times \sqrt{3} = 3\sqrt{3}$$

Negative Powers

If the power has a minus (-) sign in front of it, the base must be reciprocated. This means the number is flipped. In general: $a^{-1} = \frac{1}{a}$.

Examples: Power of -1
$2^{-1} = \frac{1}{2}$
$(\frac{1}{3})^{-1} = \frac{1}{(\frac{1}{3})} = \frac{3}{1} = 3$

If the power has a negative number that is not -1, e.g. -2, then first the number is **reciprocated** (flipped) to remove the minus sign and then the new positive power is applied.

Examples: A Negative Power
$4^{-2} = \left(\frac{1}{4}\right)^2 = \frac{1^2}{4^2} = \frac{1}{16}$
$\left(\frac{1}{5}\right)^{-3} = \left(\frac{5}{1}\right)^3 = \frac{5^3}{1^3} = 5^3 = 125$
$\left(\frac{2}{5}\right)^{-5} = \left(\frac{5}{2}\right)^5 = \frac{5^5}{2^5} = \frac{3125}{32}$

Fractional Powers (Higher Only)

The **power** can also be given in the form of a **fraction**. This creates a root with the base inside the root. The denominator of the fractional power tells us which root is applied.

Example: A fractional power with a numerator of 1
$2^{\frac{1}{2}} = \sqrt{2}$
$5^{\frac{1}{3}} = \sqrt[3]{5}$
$\left(\frac{3125}{32}\right)^{\frac{1}{5}} = \frac{\sqrt[5]{3125}}{\sqrt[5]{32}} = \frac{5}{2}$

The fractional power can be any fraction, for example: $\frac{2}{3}$. In this case more than one operation is performed on the **base**, and we need to break down our working into steps. The denominator creates a **root** first, and the numerator creates an **integer power** after:

Example: A fractional power with a numerator other than 1

$$2^{\frac{2}{3}} = (\sqrt[3]{2})^2$$

$$9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$$

$$\left(\frac{32}{3125}\right)^{\frac{4}{5}} = \left(\frac{\sqrt[5]{32}}{\sqrt[5]{3125}}\right)^4 = \left(\frac{2}{5}\right)^4 = \frac{16}{625}$$

The steps are:

1. Apply the **denominator** of the fractional power. This will create a root.
2. Simplify the **root** if possible.
3. Apply the **numerator** of the fractional power. This will create an **integer power**.
4. Simplify where possible.

Combining Negative & Fractional Powers (Higher Only)

Bringing together different types of powers, we can combine negative indices with fractional indices. For a **base** that is a fraction, the order of calculation is:

1. Flip the base (**reciprocate**) to remove the minus sign.
2. Look at the **denominator** of the power and apply the correct **root** to both the **numerator** and **denominator** of the **reciprocated** base.
3. Simplify the root where possible.
4. Look at the **numerator** of the power and apply the correct **integer power** to both the **numerator** and **denominator**.

Example: A Negative Fractional power

$$4^{-\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{2}} = \left(\frac{\sqrt{1}}{\sqrt{4}}\right) = \frac{1}{2}$$

$$\left(\frac{4}{8}\right)^{-\frac{1}{3}} = \left(\frac{8}{4}\right)^{\frac{1}{3}} = \left(\frac{\sqrt[3]{8}}{\sqrt[3]{4}}\right) = \frac{2}{\sqrt[3]{4}}$$

$$\left(\frac{32}{3125}\right)^{-\frac{1}{5}} = \left(\frac{3125}{32}\right)^{\frac{1}{5}} = \frac{\sqrt[5]{3125}}{\sqrt[5]{32}} = \frac{5}{2}$$

$$\left(\frac{6}{7}\right)^{-\frac{4}{5}} = \left(\frac{7}{6}\right)^{\frac{4}{5}} = \frac{(\sqrt[5]{7})^4}{(\sqrt[5]{6})^4}$$

Multiplying & Dividing Terms with Powers

When multiplying terms with powers, the powers are added. When dividing terms with powers, the power of the **denominator** (the second term) is subtracted from the power of the **numerator** (the first term):

Example: Multiplying Terms with Powers

$$x^2 \times x^3 = x^{2+3} = x^5$$
$$a^{\frac{3}{2}} \times a^{\frac{1}{2}} = a^{\frac{3}{2}+\frac{1}{2}} = a^2$$
$$d^{\frac{4}{5}} \times d^{-\frac{3}{5}} = d^{\frac{4}{5}+-\frac{3}{5}} = d^{\frac{1}{5}} = \sqrt[5]{d}$$

Example: Dividing Terms with Powers

$$x^8 \div x^4 = x^{8-4} = x^4$$
$$a^{\frac{7}{3}} \div a^{\frac{2}{6}} = a^{\frac{7}{3}-\frac{2}{6}} = a^{\frac{6}{3}} = a^2$$
$$d^{\frac{8}{9}} \div d^{-\frac{2}{18}} = d^{\frac{8}{9}--\frac{2}{18}} = d^{\frac{9}{9}} = d$$

Raising Terms with Powers to a Higher Power

Raising a term with an associated power to another power means we must multiply the two powers together. This applies for fractional powers too.

Example: Raising terms with powers to a power

$$(x^2)^3 = x^{2\times3} = x^6$$
$$(r^{12})^8 = r^{8\times12} = r^{96}$$
$$(y^{-3})^{\frac{3}{5}} = y^{-3\times\frac{3}{5}} = y^{-\frac{9}{5}}$$

Estimating Powers and Roots (Higher Only)

Powers can be estimated by rounding the **base** up or down to an integer and using this as an estimation. We can also look at what integers the base lies between and use those to form our answer.

Example: Estimating powers

Estimate 3.7^2

$(3.7 \simeq 4) = 4^2 = 16$

So 3.7^2 is approximately 16.

Estimate 3.7^3

3.7 is between 3 and 4, therefore

$3^3 < 3.7^3 < 4^3$

$27 < 3.7^3 < 64$

As 3.7 is closer to 4 we estimate: $3.7^3 \simeq 50$

To estimate roots, we apply a similar method but we look at the nearest square numbers and use those values to make our estimation.

Example: Estimating roots
Estimate $\sqrt{53}$
$49 < 53 < 64$
Therefore,
$\sqrt{49} < \sqrt{53} < \sqrt{64}$
$7 < \sqrt{53} < 8$
As 53 is closer to 7 than to 8 we estimate $\sqrt{53} \simeq 7.3$.

Powers, Roots and Fractional Indices - Practice Questions

1. Find $\left(\frac{3}{5}\right)^2$

2. Simplify $2x^2 \times 5x^6$

3. Simplify $(3f^5)^{\frac{9}{10}}$

4. Simplify $r^{16} \div r^5$

5. Find and simplify where possible $\left(\frac{16}{81}\right)^{-\frac{5}{4}}$ (Higher Only)

6. Estimate $\sqrt{39}$ (Higher Only)

7. Estimate 6.7^2 (Higher Only)

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.